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Note on purity of bi-ideals on semigroups

by

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1. A subsemigroup A of a semigroup S is called a bi-ideal of S if $ASA \subseteqq A$. The notion of bi-ideal was introduced by R. A. Good and D. Huges [2]. It was also a special case of the (m, n)-ideal introduced by S. Lajos [5]. Let $B(S)$ be the set of all bi-ideals of a semigroup S. We define a binary operation on $B(S)$ as follows: For X, $Y \in B(S)$,

$$XY = \{xy : x \in X \text{ and } y \in Y\} .$$

Then the product XY is a bi-ideal of S ([6] Theorem 8), and $B(S)$ is a semigroup.

A semigroup S is called regular if, for any element a of S, there exists an element x in S such that $a = axa$.

J. Luh has given in [8] the following:

PROPOSITION 1. *For a semigroup S the following conditions are equivalent*:

(1) *S is regular*;

(2) *$B(S)$ is regular*.

The following is evident:

PROPOSITION 2. *If $B(S)$ is idempotent, then it is regular.*

The present note is concerned with the problem: For what kind of semigroups does the converse of Proposition 2 hold?

A semigroup S is called intra-regular if, for any element a of S, there exist x and y in S such that $a = xa^2y$. S. Lajos proved that, for an intra-regular semigroup S, $B(S)$ is idempotent if and only if and only if B(S) is regular ([7] Theorem 36), and gave an example of a semigroup S such that $B(S)$ is regular but not idempotent ([7] Example 4). A semigroup S is called normal if $aS = Sa$ for all elements a of S ([9]). The author proved that, for a normal semigroup S, $B(S)$ is idempotent if and only if $B(S)$ is regular ([4]).

In discussing the above problem we shall introduce the notion of purity of bi-ideals of a semigroup, which is called the T-purity. It is an anologous notion of I-pure ideal of a semigroup introduced by the author [3]. In this note we shall give some properties of T-pure bi-ideals and prove that, for a semigroup S such that every bi-ideal of it is T-pure, $B(S)$ is idempotent if and only if $B(S)$ is regular. For the terminology not defined here we refer to the book by A. H. Clifford and G. B. Preston [1].

2. A bi-ideal A of a semigroup S is called T-pure if

$$A \cap xSy = xAy$$

for all elements x and y of S. A semigroup S is called T^*-pure if every bi-ideal of it is T-pure. The semigroup S itself is a trivial example of a T-pure bi-ideal of S. It is clear that a group is a T^*-pure semigroup (see, [1] p. 84).

We denote by $[a]$ the princial bi-ideal of a semigroup S generated by a in S. Then, by S. Lajos [5],

$$[a] = a \cup a^2 \cup aSa\ .$$

LEMMA 3. *For any bi-ideal A of a semigroup S, the following conditions are equivalent*:

(1) $A \cap XSY = XAY$ for all X, $Y \in B(S)$;

(2) $A \cap [x]S[y] = [x]A[y]$ for all x, $y \in S$.

Proof. It is clear that (1) implies (2). Assume that (2) holds. Let X and Y be any bi-ideals of S and $a = xsy (a \in A, x \in X, s \in S, y \in Y)$ any element of $A \cap XSY$. Then we have

$$a = xsy \in A \cap [x]S[y] = [x]A[y] \subseteqq XAY\ .$$

Thus we have

$$A \cap XSY \subseteqq XAY$$

for all X, $Y \in B(S)$. We note that (2) implies that

$$[x]A[y] \subseteqq A$$

for all $x, y \in S$. Then in order to prove that

$$XAY \subseteqq A$$

for all X, $Y \in B(S)$, let $xay (x \in X, a \in A, y \in Y)$ be any element of XAY. Then we have

$$xay \in [x]A[y] \subseteqq A$$

and so we have

$$XAY \subseteqq A\ .$$

Since the inclusion

$$XAY \subseteqq XSY$$

always holds, we have

$$XAY \subseteqq A \cap XSY$$

for all $X, Y \in B(S)$. Therefore we have

$$A \cap XSY = XAY$$

for all $X, Y \in B(S)$. Thus we obtain that (2) implies (1).

The proof is complete.

LEMMA 4. *Let A be any T-pure bi-ideal of a smigroup S. Then any one of the conditions* (1), (2) *of Lemma* 3 *holds.*

Proof. It suffices to prove that (1) of Lemma 3 holds. Let X and Y be any bi-ideals of S, and $a = xsy (a \in A, x \in X, s \in S, y \in Y)$ any element of $A \cap XSY$. Then we have

$$a = xsy \in A \cap xSy = xAy \subseteqq XAY$$

and so we have

$$A \subset XSY \subseteqq XAY .$$

Let $xay (x \in X, a \in A, y \in Y)$ be any element of XAY. Then we have

$$xay \in xAy = A \cap xSy \subseteqq A \cap XSY$$

and so we have

$$XAY \subseteqq A \cap XSY .$$

Thus we obtain that

$$A \cap XSY = XAY$$

for all $X, Y \in B(S)$, and that the T-purity of the bi-ideal A implies that (1) of Lemma 3 holds.

THEOREM 5. *For a semigroup S the following conditions are eqaivalent*:

(1) *S is T^*-pure.*

(2) *Every principal bi-ideal of S is T-pure.*

Proof. It is clear that (1) implies (2). Assume that (2) holds. Let A be any bi-ideal of S, and x and y any elements of S. Let $a = xsy$ $(a \in A, s \in S)$ be an element of $A \cap xSy$. Then we have

$$a = xsy \in [a] \cap xSy = x[a]y \subseteqq xAy$$

and so we have

$$A \cap xSy \subseteqq xAy$$

for all $x, y \in S$. We note that (2) implies

$$x[a]y \subseteqq [a]$$

for all $x, y \in S$. In order to prove that

$$xAy \subseteqq A$$

for all $x, y \in S$, let xay $(a \in A)$ be any elemnt of xAy. Then we have

$$xay \in x[a]y \subseteqq [a] \subseteqq A$$

and so we have

$$xAy \subseteqq A$$

for all $x, y \in S$. Since the inclusion

$$xAy \subseteqq xSy$$

always holds, we have

$$xAy \subseteqq A \cap xSy$$

for all $x, y \in S$. Thus we obtain that

$$A \cap xSy = xAy$$

for all $x, y \in S$. Therefore we obtain that (2) implies (1).

This complete the proof of the theorem.

3. In this section we consider some propertie of minimal T-pure bi-ideals of a semigroup.

THEOREM 6. *For any minimal bi-ideal A of a semigroup S, the following conditions are equivalent*:

(1) *A is T-pure*:

(2) *$A = xAy$ for all $x, y \in S$.*

Proof. Assume that A is T-pure. Then, for all $x, y \in S$, we have

$$xAy = A \cap xSy \subseteqq A .$$

Since xAy is a bi-ideal of S by Theorem 8 of [6], it follows from the minimality of A that

$$xAy = A .$$

Thus we obtain that (1) implies (2).

Conversely we assume that (2) holds. Then, for all $x, y \in S$, we have

$$A \cap xSy = xAy \cap xSy = xAy .$$

This means that A is T-pure. Therefore we obtain that (2) implies (1).

COROLLARY 7. *The minimal T-pure bi-ideal of a semigroup is regular.*

Proof. Let A be the minimal T-pure bi-ideal of a semigroup S, and a any element of A. Then by Theorem 6 we have

$$a \in A = aAa \ .$$

This means that A is regular.

LEMMA 8. *For any bi-ideal A of a semigoup S, the following coditions are equivalent*:

(1) $A = XAY$ *for all* $X, Y \in B(S)$;

(2) $A = [x]A[y]$ *for all* $x, y \in S$.

Proof. It is clear that (1) implies (2). Assume that (2) holds. Let X and Y be any bi-ideals of S, and x and y respectively elements of X and Y. Then we have

$$A = [x]A[y] \subseteqq XAY \ .$$

Let xay ($x \in X$, $a \in A$, $y \in Y$) be any element of XAY. Then we have

$$xay \in [x]A[y] = A$$

and so we have

$$XAY \subseteqq A \ .$$

Thus we have

$$A = XAY$$

for all $X, Y \in B(S)$. Therefore we obtain that (2) implies (1).

THEOREM 9. *Let A be any minimal T-pure bi-ideal of a semigroup S. Then any one of the conditions* (1), (2) *of Lemma* 8 *holds.*

Proof. It suffices to prove that (1) of Lemma 8 holds. Let X and Y be any bi-ideals of S. Then, since A is T-pure, it follows from Lemma 4 that

$$XAY = A \cap XSY \subseteqq A \ .$$

Since XAY is a bi-ideal of S, it follows from the minimality of A that

$$XAY = A \ .$$

Therefore we obtain that (1) of Lemma 8 holds.

4. A semigroup S is called T-pure-free if it does not properly contain any T-pure bi-ideal. In this section we give a class of a T-pure-free semigroup.

A semigroup S is called archimedean if, for any elements a and b of S, there exists a positive integer n for which

$$a^n \in SbS \ .$$

THEOREM 10. *A cancellative archimedean semigroup without idempotent is T-pure-free.*

Proof. Let A be any T-pure bi-ideal of a cancellative archmedean semigroup S without idempotent, and a and s respectively any elements of A and S. Since S is archimedean, there exist elements x and y in S and a positive integer n such that

$$a^n = xsy.$$

Since A is T-pure, we have

$$a^n = xsy \in A \cap xSy = xAy .$$

This implies that there exists an element b in A such that

$$xsy = xby .$$

Since S is cancellative, we have

$$s = b \in A$$

and so we have

$$S \subseteqq A .$$

This we obtain that

$$A = S .$$

Since A is any T-pure bi-ideal of S, this means that S is T-pure-free. This completes the proof of the theorem.

5. In this section we give our main result.

THEOREM 11. *For a T^*-pure semigroup S the following conditeons are equivalent*:

(1) *S is regular*;
(2) *$B(S)$ is regular*;
(3) *$B(S)$ is idempotent.*

Proof. By Propositions 1 and 2, it suffices to prove that (2) implies (3). We assume that $B(S)$ is regular. Let A be and bi-ideal of S. Then for some $X \in B(S)$ we have

$$A = AXA \subseteqq ASA \subseteqq A$$

and so we have

$$A = ASA .$$

Since A is T-pure, it follows from Lemma 4 that

$$XAY = A \cap XSY$$

for all X, $Y \in B(S)$. This holds for $X = Y = A$. Then we have

$$A^3 = A \cap ASA = A \cap A = A$$

and so we have

$$A = A^3 \subseteqq A^2 \subseteqq A .$$

Thus we have

$$A=A^2 .$$

Therefore we obtain that $B(S)$ is idempotent. This completes the proof of the theorem.

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